

QUANTUM GROUPS AND GENERALIZED CIRCULAR ELEMENTS

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ABSTRACT. We show that with respect to the Haar state, the joint distributions of the generators of Van Daele and Wang's free orthogonal quantum groups are modeled by free families of generalized circular elements and semicircular elements in the large (quantum) dimension limit. We also show that this class of quantum groups acts naturally as distributional symmetries of almost-periodic free Araki-Woods factors.

1. INTRODUCTION

There are intriguing connections between the representation theory of certain classes of compact matrix groups, and independent Gaussian structures in probability theory. For instance, if one considers the N^2 matrix elements $\{u_{ij}\}_{1 \leq i, j \leq N}$ of the fundamental representation of the $N \times N$ orthogonal group $O_N = O_N(\mathbb{R})$ on the Hilbert space \mathbb{C}^N , then it is well known that the joint moments of these variables with respect to the Haar probability measure are approximated by an independent and identically distributed, mean zero, variance $\frac{1}{N}$ family of real Gaussian random variables in the large N limit (see for example [10]). Intimately related to this asymptotic Gaussianity result is the celebrated theorem of Freedman [11, 10], which says that an infinite sequence $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ of real-valued random variables is a conditionally independent centered Gaussian family with common variance if and only if the sequence is *rotatable*: i.e., for each $N \in \mathbb{N}$, the joint distribution of the N -dimensional truncation $\mathbf{x}_N = (x_n)_{1 \leq n \leq N}$ of \mathbf{x} is invariant under rotations by O_N .

When one replaces the family of orthogonal groups $\{O_N\}_{N \in \mathbb{N}}$ by the unitary groups $\{U_N\}_{N \in \mathbb{N}}$, analogous results are known to hold where one replaces real Gaussian random variables by their complex-valued counterparts. The key ingredient for the above results is a certain asymptotic orthonormality property for canonical generators (weighted Brauer diagrams, in fact) of the spaces of intertwiners between the tensor powers of the fundamental representations of these groups in the large rank limit. This asymptotic feature of the representation theory is most concisely expressed via the so-called Weingarten calculus developed in [6, 7], with origins in the pioneering work of Weingarten [21] on the asymptotics of unitary matrix integrals. For a broad treatment of probabilistic symmetries, we refer to the text [12].

Within the framework of operator algebras and non-commutative geometry, *compact quantum groups* provide a vast and rich generalization of the theory of compact groups. The operator algebraic theory of compact quantum groups was pioneered by Woronowicz (see [22, 23] for instance) and has led recently to many interesting examples and developments in the theory of operator algebras.

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One can ask non-commutative probabilistic questions about compact quantum groups, because every compact quantum group \mathbb{G} admits a natural analogue of the Haar probability measure (the Haar state). The last decade or so has seen a flurry of activity in this direction, particularly for free quantum groups and Voiculescu's free probability theory. For instance, the free orthogonal quantum groups O_N^+ and free unitary quantum groups U_N^+ discovered by Wang [20] have interesting probabilistic statements that show a deep parallel with the aforementioned classical results for O_N and U_N . Most notable for our purposes are the results of Banica-Collins [3] and Curran [8]. Banica and Collins show that the rescaled matrix elements $\{\sqrt{N}u_{ij}\}_{1 \leq i, j \leq N}$ of the fundamental representation of O_N^+ (respectively U_N^+) converge in joint distribution to a freely independent family of standard-mean zero, variance one-semicircular (respectively circular) elements in a free group factor. Curran provides a free probability analogue of Freedman's rotatability theorem: An infinite sequence $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ of self-adjoint non-commutative random variables in a W^* -probability space (M, φ) is *quantum rotatable* if and only if there exists a W^* -subalgebra $B \subseteq M$ and a φ -preserving conditional expectation $E : M \rightarrow B$ such that \mathbf{x} is an identically distributed family of mean zero semicircular elements that is free with amalgamation over B . A similar result for U_N^+ is obtained in [8].

In this paper, we consider similar non-commutative probabilistic questions for a broad class of compact quantum groups introduced by Van Daele and Wang [18] that generalizes the construction of U_N^+ and O_N^+ (which they called *universal quantum groups*). To define such a universal quantum group, one uses an invertible matrix $F \in \text{GL}_N(\mathbb{C})$ to deform the defining algebraic relations for the quantum groups O_N^+ and U_N^+ , yielding a pair of new compact matrix quantum groups called O_F^+ and U_F^+ . When $F = 1$ (the $N \times N$ identity matrix), we recover O_N^+ and U_N^+ as special cases (see Section 3 for precise definitions). We follow recent literature conventions and refer to O_F^+ as *free orthogonal quantum groups* and U_F^+ as *free unitary quantum groups* (both with parameter matrix F).

The quantum groups $\mathbb{G} = O_F^+, U_F^+$ (at least when F is not a multiple of a unitary matrix) are especially interesting because their Haar states are non-tracial and because the corresponding quantum group von Neumann algebras $L^\infty(\mathbb{G})$ are known to be type III factors in many cases (see [9, 17]).

The main result of this paper is that in this far more general (possibly non-tracial) setting, asymptotic freeness still emerges in the large rank limit. In theorems 5.1, 5.2 and 5.4, we show that the joint distribution of the (suitably rescaled) matrix elements of the fundamental representation of the quantum group O_F^+ can be approximated by a freely independent family of non-commutative random variables consisting of semicircular elements and Shlyakhtenko's *generalized circular elements* [14], which is built in a natural way from the initial data $F \in \text{GL}_N(\mathbb{C})$. Generalized circular elements are non-tracial deformations of Voiculescu's circular elements, and they arise as canonical generators of free Araki-Woods factors [14]. Since free Araki-Woods factors are the natural non-tracial (or type III) deformations of the free group factors, our asymptotic freeness results for O_F^+ provide a satisfactory generalization of the tracial asymptotic freeness results of [3]. We also remark in Section 5.3 how similar asymptotic freeness results can be obtained for the free unitary quantum groups U_F^+ .

Our proofs of asymptotic freeness results in O_F^+ and U_F^+ follow the general outline of the earlier work of [3, 7], and we use a modified version of the Weingarten calculus for our situation. In our case, the formulas become a bit more unwieldy, a consequence of the extra

parameters that arise from the non-trivial matrix $F \in \text{GL}_N(\mathbb{C})$. On the other hand, there is one significant and interesting difference between our (non-tracial) setting and the earlier asymptotic (free) independence results on groups and quantum groups where the Haar state is tracial. In the case of O_F^+ and U_F^+ , $F \in \text{GL}_N(\mathbb{C})$, we find that the error in approximation of joint moments by free variables is of order $O\left((\text{Tr}(F^*F))^{-1}\right)$, a bound that is in many cases much smaller than the traditional bound given by $O\left(\frac{1}{N}\right)$ in the classical case. This fact allows us to observe, for example, asymptotic freeness results in a fixed dimension N , by considering families of quantum groups $\{O_F^+\}_{F \in \Lambda \subset \text{GL}_N(\mathbb{C})}$ where the *quantum dimension* $\text{Tr}(F^*F)$ tends to infinity (cf. Theorem 5.4).

Based on our non-tracial asymptotic freeness results described above, together with Curran's work on quantum rotatability [8], it now becomes natural to ask whether the free quantum groups O_F^+ act non-trivially on free Araki-Woods factor in a free-quasi-free state-preserving way. In Section 6, we answer this question in the affirmative, and as a result we observe that almost-periodic free Araki-Woods factors admit a wealth of quantum symmetries. A future goal of the authors is to find a suitable "type III" version of Freedman's theorem adapted to O_F^+ , U_F^+ , and free Araki-Woods factors. After a first version of this paper appeared, it was pointed out to the authors that the main result of Section 6, namely Theorem 6.5, can also be obtained as a special case of a very general result of S. Vaes [16, Proposition 3.1].

We finish this section with an outline of paper's organization. Section 2 discusses some preliminaries on quantum groups and free probability that are required. Section 3 defines the quantum groups O_F^+ and U_F^+ and gives Weingarten-type formulas for joint moments of the generators of O_F^+ with respect to the Haar state. Section 5 gives various asymptotic freeness results for O_F^+ , and includes a remark on how to extend our results on O_F^+ to some of their unitary counterparts U_F^+ . Finally Section 6 considers O_F^+ as quantum symmetries of almost-periodic free Araki-Woods factors. This is achieved by associating to each O_F^+ a canonical free family of generalized circular elements whose joint distribution is invariant under quantum rotations by O_F^+ .

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2. PRELIMINARIES

In this section we briefly review some concepts from free probability theory and compact quantum group theory. For more details, we refer the reader to [13] for free probability and [15, 23] for quantum groups.

2.1. Non-commutative probability spaces and free independence. A *non-commutative probability space (NCPS)* is a pair (A, φ) , where A is a unital C^* -algebra, and $\varphi : A \rightarrow \mathbb{C}$ is a state (i.e. a linear functional such that $\varphi(1_A) = 1$ and $\varphi(a^*a) \geq 0$ for all $a \in A$). Elements $a \in A$ are called *random variables*. Given a family of random variables $X = \{x_r\}_{r \in \Lambda} \subset (A, \varphi)$, the *joint distribution* of X is the collection of all *joint $*$ -moments*

$$\{\varphi(P((x_r)_{r \in \Lambda})) : P \in \mathbb{C}\langle t_r, t_r^* : r \in \Lambda \rangle\},$$

where $\mathbb{C}\langle t_r, t_r^* : r \in \Lambda \rangle$ is the unital $*$ -algebra of non-commutative polynomials in the variables $\{t_r\}_{r \in \Lambda}$, equipped with anti-linear involution $t_r \mapsto t_r^*$. Given another family of random variables $Y = \{y_r\}_{r \in \Lambda}$ in a NCPS (B, ψ) , we say that X and Y are *identically distributed* if $\varphi(P((x_r)_{r \in \Lambda})) = \psi(P((y_r)_{r \in \Lambda}))$ for all $P \in \mathbb{C}\langle t_r, t_r^* : r \in \Lambda \rangle$.

Let (A, φ) be a NCPS. A family of $*$ -subalgebras $\{A_r\}_{r \in \Lambda}$ of A is said to be *freely independent* (or simply *free*) if the following condition holds: for any choice of indices $r(1) \neq r(2), r(2) \neq r(3), \dots, r(k-1) \neq r(k) \in \Lambda$ and any choice of centered random variables $x_{r(j)} \in A_{r(j)}$ (i.e., $\varphi(x_{r(j)}) = 0$), we have the equality

$$\varphi(x_{r(1)}x_{r(2)} \dots x_{r(k)}) = 0.$$

A family of random variables $X = \{x_r\}_{r \in \Lambda} \subset (A, \varphi)$ is said to be *free* if the family of unital $*$ -subalgebras

$$\{A_r\}_{r \in \Lambda}; \quad A_r := \text{alg}(1, x_r, x_r^*),$$

is free in the above sense. Let $S_\alpha = \{x_r^{(\alpha)}\}_{r \in \Lambda} \subset (A_\alpha, \varphi_\alpha)$ be a net of families of random variables and $S = \{x_r\}_{r \in \Lambda} \in (A, \varphi)$ be another family of random variables. We say that S_α *converges to S (or $S_\alpha \rightarrow S$) in distribution* if, for any non-commutative polynomial $P \in \mathbb{C}\langle X_r : r \in \Lambda \rangle$, we have

$$\lim_{\alpha} \varphi_\alpha(P(S_\alpha)) = \varphi(P(S)).$$

2.2. Fock spaces, semicircular elements, and generalized circular elements. Let H be a complex Hilbert space. The *full fock space* is the Hilbert space

$$\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} H^{\otimes n},$$

where we put $H^{\otimes 0} := \mathbb{C}\Omega$, where Ω is a fixed unit vector, called the *vacuum vector*. The *vacuum expectation* is the state $\varphi_\Omega : \mathcal{B}(\mathcal{F}(H)) \rightarrow \mathbb{C}$ given by $\varphi_\Omega(x) = \langle \Omega | x \Omega \rangle$, $x \in \mathcal{B}(\mathcal{F}(H))$.

For each $\xi \in H$, we define the *left creation operator* $\ell(\xi) \in \mathcal{B}(\mathcal{F}(H))$ by

$$\ell(\xi)\Omega = \xi, \quad \ell(\xi)\eta = \xi \otimes \eta \quad (\eta \in H^{\otimes n}, n \geq 1).$$

Note that $\|\ell(\xi)\|_{\mathcal{B}(\mathcal{F}(H))} = \|\xi\|_H$. Given a NCPS (A, φ) , a *(standard) semicircular element* is a self-adjoint random variable $x \in A$ with the same distribution as $s(\xi) := \ell(\xi) + \ell(\xi)^* \in (\mathcal{B}(\mathcal{F}(H)), \varphi_\Omega)$, where $\xi \in H$ is a unit vector. Given $\alpha, \beta \in \mathbb{R}^+$, a random variable $x \in (A, \varphi)$ is called an (α, β) -*generalized circular element* if it has the same distribution as the element $\alpha\ell(\xi) + \beta\ell(\eta)^* \in (\mathcal{B}(\mathcal{F}(H)), \varphi_\Omega)$, where ξ, η are orthonormal vectors in H . One can readily verify that for an (α, β) -generalized circular element x , one has

$$\varphi(x^*x) = \alpha^2 \quad \& \quad \varphi(xx^*) = \beta^2,$$

and this information completely determines the $*$ -moments of x with respect to φ . We will call the numbers α^2 and β^2 the left and right variances of x , respectively.

Next, we want state a well known theorem which gives a combinatorial characterization of the joint distribution of a free semicircular or generalized circular family. To do this, we first need some notation concerning non-crossing partitions that will be used below and throughout the remainder of the paper.

Notation 2.1. Let $k \in \mathbb{N}$ and denote by $[k]$ the ordered set $\{1, \dots, k\}$.

- (1) The lattice of (non-crossing) partitions of $[k]$ will be denoted by $\mathcal{P}(k)$ (resp. $NC(k)$), and the standard partial order on both lattices will be denoted by \leq .

- (2) If $\pi \in \mathcal{P}(k)$ partitions $[k]$ into r disjoint, non-empty subsets $\mathcal{V}_1, \dots, \mathcal{V}_r$ (called *blocks*), we write $|\pi| = r$ and say that π *has r blocks*.
- (3) Given a function $i : [k] \rightarrow \Lambda$, we denote by $\ker i$ the element of $\mathcal{P}(k)$ whose blocks are the equivalence classes of the relation

$$s \sim_{\ker i} t \iff i(s) = i(t).$$

Note that if $\pi \in \mathcal{P}(k)$, then $\pi \leq \ker i$ is equivalent to the condition that whenever s and t are in the same block of π , $i(s)$ must equal $i(t)$ (i.e the function $i : [k] \rightarrow \Lambda$ is constant on the blocks of π).

- (4) Elements of $\mathcal{P}(k)$ which partition $[k]$ into subsets with exactly two elements are called *pairings* and the set of pairings of $[k]$ is denoted by $\mathcal{P}_2(k)$. We also write $NC_2(k) = \mathcal{P}_2(k) \cap NC(k)$. If k is odd, we of course have $\mathcal{P}_2(k) = NC_2(k) = \emptyset$.
- (5) Given $\pi \in \mathcal{P}_2(k)$ and $s, t \in [k]$, we will always write $(s, t) \in \pi$ if $\{s, t\}$ is a block of π and $s < t$.
- (6) Let $\epsilon : [k] \rightarrow \{1, *\}$ be a function. We let $NC_2^\epsilon(k) \subset NC_2(k)$ be the subset of all non-crossing pairings such that

$$\forall (s, t) \in \pi \implies \epsilon(s) \neq \epsilon(t).$$

Theorem 2.2 (See Chapters 7 and 15 of [13]). *Let $X = (x_r)_{r \in \Lambda}$ be a family of random variables in a NCPS (A, φ) .*

- (1) *If $x_r = x_r^*$ for each $r \in \Lambda$, then X is a free family of standard semicircular variables if and only if*

$$\begin{aligned} \varphi(x_{r(1)} \dots x_{r(k)}) &= \sum_{\substack{\pi \in NC_2(k) \\ \ker r \geq \pi}} \prod_{(s,t) \in \pi} \varphi(x_{r(s)} x_{r(t)}) \\ &= |\{\pi \in NC_2(k) : \ker r \geq \pi\}| \quad (k \in \mathbb{N}, r : [k] \rightarrow \Lambda). \end{aligned}$$

- (2) *Let $(\alpha_r, \beta_r)_{r \in \Lambda} \subset \mathbb{R}^+ \times \mathbb{R}^+$. Then X is a free family of (α_r, β_r) -generalized circular elements if and only if*

$$\varphi(x_{r(1)}^{\epsilon(1)} \dots x_{r(k)}^{\epsilon(k)}) = \sum_{\substack{\pi \in NC_2^\epsilon(k) \\ \ker r \geq \pi}} \prod_{(s,t) \in \pi} \varphi(x_{r(s)}^{\epsilon(s)} x_{r(t)}^{\epsilon(t)}) \quad (k \in \mathbb{N}, r : [k] \rightarrow \Lambda, \epsilon : [k] \rightarrow \{1, *\}).$$

2.3. Free Araki-Woods factors and generalized circular elements. Let $H_{\mathbb{R}}$ be a real separable Hilbert space and let (U_t) be an orthogonal representation of \mathbb{R} on $H_{\mathbb{R}}$. Let $H = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified Hilbert space. If A is the infinitesimal generator of (the extension of) U_t on H (i.e., $U_t = A^{it}$), then it follows that the map $j : H_{\mathbb{R}} \hookrightarrow H$ defined by $j(\xi) = \left(\frac{2}{A^{-1}+1}\right)^{1/2} \xi$ is an isometric embedding of $H_{\mathbb{R}}$ into H [14]. Let $K_{\mathbb{R}} = j(H_{\mathbb{R}})$, then we have $K_{\mathbb{R}} \cap iK_{\mathbb{R}} = \{0\}$ and $K_{\mathbb{R}} + iK_{\mathbb{R}}$ is dense in H . The *free Araki-Woods factor* is the von Neumann algebra

$$\Gamma(H_{\mathbb{R}}, U_t)'' = W^*(\ell(\xi) + \ell(\xi)^* : \xi \in K_{\mathbb{R}}) \subseteq \mathcal{B}(\mathcal{F}(H)).$$

The restriction of the vacuum expectation $\varphi_{\Omega} = \langle \Omega | \cdot \Omega \rangle$ on $\mathcal{B}(\mathcal{F}(H))$ to $\Gamma(H_{\mathbb{R}}, U_t)''$ is always a faithful normal state, and turns $(\Gamma(H_{\mathbb{R}}, U_t)'', \varphi_{\Omega})$ into a non-commutative probability space.

We recall from [14] that U_t is the trivial representation if and only if $\Gamma(H_{\mathbb{R}}, U_t)'' \cong L(\mathbb{F}_{\dim H_{\mathbb{R}}})$, the von Neumann algebra generated by the left regular representation of the free group on $\dim H_{\mathbb{R}}$ generators. Otherwise, $\Gamma(H_{\mathbb{R}}, U_t)''$ is a type III factor.

Free Araki-Woods factors arise naturally when one considers free families of generalized circular elements that we introduced earlier. More precisely, we have the following theorem, which follows easily from the results in [14, Section 6].

Theorem 2.3 ([14]). *Let $X = (x_r)_{r \in \Lambda}$ be a free family of (α_r, β_r) -generalized circular elements in a non-commutative probability space (A, φ) and let $0 < \lambda_r = \min\{\alpha\beta^{-1}, \beta\alpha^{-1}\} \leq 1$. Then there is a state-preserving $*$ -isomorphism $(W^*(X), \varphi) \cong (\Gamma(H_{\mathbb{R}}, U_t)'', \varphi_{\Omega})$, where U_t is the almost-periodic orthogonal representation acting on the Hilbert space $H_{\mathbb{R}} = \bigoplus_{r \in \Lambda} \mathbb{R}^2$ given by*

$$U_t = \bigoplus_{r \in \Lambda} R_{\lambda_r}(t) \quad \text{where} \quad R_{\lambda_r}(t) = \begin{bmatrix} \cos(t \log \lambda_r) & -\sin(t \log \lambda_r) \\ \sin(t \log \lambda_r) & \cos(t \log \lambda_r) \end{bmatrix}.$$

Moreover, every free Araki-Woods factor $\Gamma(H_{\mathbb{R}}, U_t)''$ arising from an almost-periodic representation U_t arises in this fashion.

2.4. Compact quantum groups. A compact quantum group is a pair $\mathbb{G} = (C(\mathbb{G}), \Delta)$ where $C(\mathbb{G})$ is a unital C^* -algebra and $\Delta : C(\mathbb{G}) \rightarrow C(\mathbb{G}) \otimes C(\mathbb{G})$ is a unital $*$ -homomorphism satisfying

$$(\iota \otimes \Delta)\Delta = (\Delta \otimes \iota)\Delta \quad (\text{coassociativity})$$

$$[\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))] = [\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)] = C(\mathbb{G}) \otimes C(\mathbb{G}) \quad (\text{non-degeneracy}),$$

where $[S]$ denotes the norm-closed linear span of a subset $S \subset C(\mathbb{G}) \otimes C(\mathbb{G})$. Here and in the rest of the paper, the symbol \otimes will denote the minimal tensor product of C^* -algebras, $\overline{\otimes}$ will denote the spatial tensor product of von Neumann algebras, and \odot will denote the algebraic tensor product of complex associative algebras. The homomorphism Δ is called a *coproduct*.

For any compact quantum group $\mathbb{G} = (C(\mathbb{G}), \Delta)$, there exists a unique *Haar state* $h_{\mathbb{G}} : C(\mathbb{G}) \rightarrow \mathbb{C}$ which satisfies the following left and right Δ -invariance property, for all $a \in C(\mathbb{G})$:

$$(2.1) \quad (h_{\mathbb{G}} \otimes \iota)\Delta(a) = (\iota \otimes h_{\mathbb{G}})\Delta(a) = h_{\mathbb{G}}(a)1.$$

Note that in general $h = h_{\mathbb{G}}$ is not faithful on $C(\mathbb{G})$. In any case, we can construct a GNS representation $\pi_h : C(\mathbb{G}) \rightarrow \mathcal{B}(L^2(\mathbb{G}))$, where $L^2(\mathbb{G})$ is the Hilbert space obtained by separation and completion of $C(\mathbb{G})$ with respect to the sesquilinear form $\langle a|b \rangle = h(a^*b)$, and π_h is the natural extension to $L^2(\mathbb{G})$ of the left multiplication action of $C(\mathbb{G})$ on itself. The *von Neumann algebra of \mathbb{G}* is given by

$$L^\infty(\mathbb{G}) = \pi_h(C(\mathbb{G}))'' \subseteq \mathcal{B}(L^2(\mathbb{G})).$$

We note that Δ_r extends to an injective normal $*$ -homomorphism $\Delta_r : L^\infty(\mathbb{G}) \rightarrow L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$, and the Haar state on $C(\mathbb{G})$ lifts to a faithful normal Δ_r -invariant state on $L^\infty(\mathbb{G})$.

Let H be a finite dimensional Hilbert space and $U \in \mathcal{B}(H) \otimes C(\mathbb{G})$ be invertible (unitary). Then U is called a (*unitary*) *representation* of \mathbb{G} if, following the leg numbering convention,

$$(2.2) \quad (\iota \otimes \Delta)U = U_{12}U_{13}.$$

If we fix an orthonormal basis of H , we can identify U with an invertible matrix $U = [u_{ij}] \in M_N(C(\mathbb{G}))$ and (2.2) means exactly that

$$\Delta(u_{ij}) = \sum_{k=1}^N u_{ik} \otimes u_{kj} \quad (1 \leq i, j \leq N).$$

Of course the unit $1 \in C(\mathbb{G})$ is always a representation of \mathbb{G} , called the *trivial representation*.

Let $U \in \mathcal{B}(H_1) \otimes C(\mathbb{G})$ and $V \in \mathcal{B}(H_2) \otimes C(\mathbb{G})$ be two representations of \mathbb{G} . An *intertwiner* between U and V is a bounded linear map $T : H_1 \rightarrow H_2$ such that $(T \otimes \iota)U = V(T \otimes \iota)$. The Banach space of all such intertwiners is denoted by $\text{Hom}_{\mathbb{G}}(U, V)$. When $U = 1$ is the trivial representation, we write $\text{Hom}_{\mathbb{G}}(U, V) = \text{Fix}(V) \subset H_2$, and call $\text{Fix}(V)$ the *space of fixed vectors* for V . If there exists an invertible (unitary) intertwiner between U and V , they are said to be *(unitarily) equivalent*. A representation is said to be irreducible if its only self-intertwiners are the scalar multiples of the identity map. It is known that each irreducible representation of \mathbb{G} is finite dimensional and every finite dimensional representation is equivalent to a unitary representation. In addition, every unitary representation is unitarily equivalent to a direct sum of irreducible representations.

A compact quantum group \mathbb{G} is called a *compact matrix quantum group* if there exists a finite dimensional unitary representation $U = [u_{ij}] \in M_N(C(\mathbb{G}))$ whose matrix elements generate $C(\mathbb{G})$ as a C^* -algebra. Such a representation U is called a *fundamental representation* of \mathbb{G} . In this case, we note that the Haar state h is faithful when restricted to the dense unital $*$ -subalgebra $\text{Pol}(\mathbb{G}) \subseteq C(\mathbb{G})$ generated by $\{u_{ij}\}_{1 \leq i, j \leq N}$.

3. THE FREE QUANTUM GROUPS O_F^+ AND U_F^+

In this section we recall the definition of the free orthogonal and unitary quantum groups O_F^+ and U_F^+ , introduced by Van Daele and Wang in [18].

Notation 3.1. Given a complex $*$ -algebra A and a matrix $X = [x_{ij}] \in M_N(A)$, we denote by \bar{X} the matrix $[x_{ij}^*] \in M_N(A)$.

Definition 3.2 ([18]). Let $N \geq 2$ be an integer and let $F \in \text{GL}_N(\mathbb{C})$.

- (1) The *free unitary quantum group* U_F^+ (with parameter matrix F) is the compact quantum group given by the universal C^* -algebra

$$(3.1) \quad C(U_F^+) = C^*(\{v_{ij}\}_{1 \leq i, j \leq N} \mid U = [v_{ij}] \text{ is unitary \& } F\bar{U}F^{-1} \text{ is unitary}),$$

together with coproduct $\Delta : C(U_F^+) \rightarrow C(U_F^+) \otimes C(U_F^+)$ given by

$$\Delta(v_{ij}) = \sum_{k=1}^N v_{ik} \otimes v_{kj} \quad (1 \leq i, j \leq N).$$

- (2) Let $c = \pm 1$ and assume that $F\bar{F} = c1$. The *free orthogonal quantum group* O_F^+ (with parameter matrix F) is the compact quantum group given by the universal C^* -algebra

$$(3.2) \quad C(O_F^+) = C^*(\{u_{ij}\}_{1 \leq i, j \leq N} \mid U = [u_{ij}] \text{ is unitary \& } U = F\bar{U}F^{-1}),$$

together with coproduct $\Delta : C(O_F^+) \rightarrow C(O_F^+) \otimes C(O_F^+)$ given by

$$\Delta(u_{ij}) = \sum_{k=1}^N u_{ik} \otimes u_{kj} \quad (1 \leq i, j \leq N).$$

Remark 3.3. The coproduct Δ is defined so that the matrix of generators $U = [u_{ij}]$ is always a fundamental representation of the compact matrix quantum groups U_F^+ and O_F^+ , respectively.

Remark 3.4. Note that the above definition for O_F^+ makes sense for any $F \in GL_N(\mathbb{C})$. The additional condition $F\bar{F} = \pm 1$ is equivalent to the requirement that U is always an irreducible representation of O_F^+ . Indeed, Banica [1] showed that U is irreducible if and only if $F\bar{F} = \pm \lambda 1$ ($\lambda > 0$), and moreover we clearly have $O_F^+ = O_{\lambda^{-1/2}F}^+$.

We remark that for our asymptotic freeness results, our assumption that $F\bar{F} = \pm I$ is not a major restriction. Indeed, by a result of Wang [19, Section 6], O_F^+ for generic $F \in GL_N(\mathbb{C})$ can be decomposed into a free product of finitely many quantum groups $O_{F_i}^+$ and $U_{P_k}^+$ with F_i, P_k invertible matrices and $F_i\bar{F}_i = \pm 1$.

For the remainder of the paper we will deal mostly with the free orthogonal quantum groups O_F^+ . Later on in Section 5.3 we indicate how to extend some of our orthogonal results to the unitary case.

3.1. Canonical F -matrices for O_F^+ . Let $c \in \{\pm 1\}$ and let $F \in GL_N(\mathbb{C})$ be such that $F\bar{F} = c1$. In [5], it is shown that if $c = 1$, then there is an integer $0 \leq k \leq N/2$, a non-decreasing sequence $\rho = (\rho_i)_{i=1}^k \in (0, 1)^k$, and a unitary $w \in U_N$ such that

$$(3.3) \quad F_\rho^{(+1)} := w^t F w = \begin{pmatrix} 0 & D_k(\rho) & 0 \\ D_k(\rho)^{-1} & 0 & 0 \\ 0 & 0 & 1_{N-2k} \end{pmatrix},$$

where $D_k(\rho)$ denotes the $k \times k$ diagonal matrix with diagonal entries given by the k -tuple ρ .

On the other hand if $c = -1$, then by [5] we must have $N = 2k$ is even and there exists a non-decreasing sequence $\rho = (\rho_i)_{i=1}^k \in (0, 1]^k$ and a unitary $w \in U_N$ such that

$$(3.4) \quad F_\rho^{(-1)} := w^t F w = \begin{pmatrix} 0 & D_k(\rho) \\ -D_k(\rho)^{-1} & 0 \end{pmatrix}.$$

Remark 3.5. Note that the Kac type quantum groups O_N^+ correspond to the case $F = 1_N$, which is the canonical deformation matrix $F_\rho^{(+1)}$ with $k = 0$.

According to [5], given two matrices $F_i \in GL_{N_i}(\mathbb{C})$ such that $F_i\bar{F}_i = c_i 1$, the two free orthogonal quantum groups $O_{F_1}^+$ and $O_{F_2}^+$ are isomorphic if and only if $N_1 = N_2$, $c_1 = c_2$, and $F_2 = v F_1 v^t$ for some unitary matrix $v \in U_{N_1}$. The corresponding equivalence relation on such matrices has fundamental domain given by all matrices of the form $F_\rho^{(\pm 1)}$. As a consequence, we call such matrices $F_\rho^{(\pm 1)}$ *canonical F -matrices*. The canonical F -matrices yield the most natural coordinate system in which to represent the isomorphism equivalence class of any given O_F^+ .

3.2. Integration over O_F^+ . In this section, we consider the problem of evaluating arbitrary monomials in the generators $\{u_{ij}\}_{1 \leq i, j \leq N}$ of $C(O_F^+)$ with respect to the Haar state $h_{O_F^+}$.

Notation 3.6. Fix an orthonormal basis $\{e_i\}_{i=1}^N$ for \mathbb{C}^N and $F \in GL_N(\mathbb{C})$. Define

$$\xi = \sum_{i=1}^N e_i \otimes e_i \quad \text{and} \quad \xi^F = (\text{id} \otimes F)\xi = \sum_{i=1}^N e_i \otimes F e_i.$$

For each $l \in \mathbb{N}$, $\pi \in NC_2(2l)$, and $i : [2l] \rightarrow [N]$ define

$$\delta_\pi^F(i) = \prod_{(s,t) \in \pi} F_{i(t)i(s)},$$

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and put

$$\xi_\pi^F = \sum_{i:[2l] \rightarrow [N]} \delta_\pi^F(i) e_{i(1)} \otimes e_{i(2)} \otimes \dots \otimes e_{i(2l)} \in (\mathbb{C}^N)^{\otimes 2l}.$$

For the purposes of integrating monomials over O_F^+ with respect to the Haar state, we are interested in the l -th tensor power of the fundamental representation $U = [u_{ij}]$ of O_F^+ ,

$$U^{\oplus l} := [u_{i(1)j(1)} \dots u_{i(l)j(l)}] \in \mathcal{B}((\mathbb{C}^N)^{\otimes l}) \otimes C(O_F^+).$$

$U^{\oplus l}$ is evidently a representation of the quantum group O_F^+ , and the following theorem of Banica describes the space of fixed vectors of these higher tensor powers of U .

Theorem 3.7 ([1]). *Let $N \geq 2$, $c \in \{\pm 1\}$ and $F \in GL_N(\mathbb{C})$ be such that $F\bar{F} = c1$. Then for each $l \in \mathbb{N}$,*

$$\text{Fix}(U^{\oplus 2l+1}) = \{0\},$$

and

$$\text{Fix}(U^{\oplus 2l}) = \text{span}\{\xi_\pi^F : \pi \in NC_2(2l)\}.$$

Moreover, $\{\xi_\pi^F\}_\pi$ is a linear basis for $\text{Fix}(U^{\oplus 2l})$.

With the preceding theorem at hand, we now use the Weingarten calculus to describe the Haar state on O_F^+ in terms of the Gram matrices associated to the bases $\{\xi_\pi^F\}_{\pi \in NC_2(2l)}$ of $\text{Fix}(U^{\oplus 2l})$. For each $l \in \mathbb{N}$, define an $|NC_2(2l)| \times |NC_2(2l)|$ matrix $G_{2l,F} = [G_{2l,F}(\pi, \sigma)]_{\pi, \sigma \in NC_2(2l)}$ by

$$G_{2l,F}(\pi, \sigma) = \langle \xi_\pi^F | \xi_\sigma^F \rangle \quad (\pi, \sigma \in NC_2(2l)).$$

Theorem 3.8. *Let $N \geq 2$, $c \in \{\pm 1\}$ and $F \in GL_N(\mathbb{C})$ be such that $F\bar{F} = c1$. Set $N_F := \text{Tr}(F^*F)$. Then for any $l \geq 1$, $G_{2l,F}$ is an invertible matrix and*

$$G_{2l,F}(\pi, \sigma) = c^{l+|\pi \vee \sigma|} N_F^{|\pi \vee \sigma|} \quad (\pi, \sigma \in NC_2(2l)),$$

where $\pi \vee \sigma$ denotes the join of π and σ in the lattice $\mathcal{P}(2l)$.

Proof. The first assertion follows from Theorem 3.7. For the second assertion, fix $\pi, \sigma \in NC_2(2l)$ and let $\pi \vee \sigma = \{\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_m\}$, where each \mathcal{V}_a is a block of $\pi \vee \sigma$. Then we have

$$G_{2l,F}(\pi, \sigma) = \sum_{i:[2l] \rightarrow [N]} \overline{\delta_\pi^F(i)} \delta_\sigma^F(i) = \prod_{a=1}^m \left(\sum_{i:\mathcal{V}_a \rightarrow [N]} \overline{\delta_{\pi|_{\mathcal{V}_a}}^F(i)} \delta_{\sigma|_{\mathcal{V}_a}}^F(i) \right).$$

From the above equation we see that $G_{2l,F}(\pi, \sigma)$ is a multiplicative function of the blocks of $\pi \vee \sigma$, and therefore it suffices to prove the theorem when $\pi \vee \sigma = 1_{2l}$.

To prove this special case of the theorem, we proceed by induction on l : If $2l = 2$, then $G_{2l,F} = \langle \xi^F | \xi^F \rangle = \text{Tr}(F^*F) = N_F = c^2 N_F$. Now assume $l \geq 2$ and that the desired result is true for all $\pi', \sigma' \in NC_2(2l-2)$ with $\pi' \vee \sigma' = 1_{2l-2}$. Fix $\pi, \sigma \in NC_2(2l)$ such that $\pi \vee \sigma = 1_{2l}$. Since π is non-crossing, we can fix an interval $\{r, r+1\}$ in π and let $\{a, r\}$, $\{b, r+1\}$ be the corresponding (unordered) pairs of σ that connect to r and $r+1$. (Note that σ does not pair r and $r+1$ because $|\pi \vee \sigma| = 1$ and $l \geq 2$.) Now let $\pi' \in NC_2(2l-2)$ be the pairing obtained by deleting the block $\{r, r+1\}$ from π and let $\sigma' \in NC_2(2l-2)$ be the pairing obtained by deleting the points $r, r+1$ from σ and pairing a and b . Note that by construction, $\pi' \vee \sigma' = 1_{2l-2}$.

Using the readily verified identities

$$c\xi^F = (\iota \otimes (\xi^F)^* \otimes \iota)(\xi^F \otimes \xi^F) = ((\xi^F)^* \otimes \iota)(\iota \otimes \xi^F \otimes \iota)\xi^F = (\iota \otimes (\xi^F)^*)(\iota \otimes \xi^F \otimes \iota)\xi^F,$$

it easily follows that $G_{2l,F}(\pi, \sigma) = \langle \xi_\pi^F | \xi_\sigma^F \rangle = c \langle \xi_{\pi'}^F | \xi_{\sigma'}^F \rangle$. We then have from our induction assumption that

$$G_{2l,F}(\pi, \sigma) = c(c^{l-1+|\pi' \vee \sigma'|} N_F) = c^{l+1} N_F.$$

□

For each $l \in \mathbb{N}$, denote by $W_{2l,F}$ the matrix inverse of $G_{2l,F}$. In the following theorem we give a Weingarten-type moment formula for the Haar state on O_F^+ (compare with [3, 4]).

Theorem 3.9. *For each pair of multi-indices $i, j : [l] \rightarrow [N]$, we have*

$$h_{O_F^+}(u_{i(1)j(1)} u_{i(2)j(2)} \cdots u_{i(l)j(l)}) = 0$$

if l is odd, and otherwise

$$h_{O_F^+}(u_{i(1)j(1)} u_{i(2)j(2)} \cdots u_{i(l)j(l)}) = \sum_{\pi, \sigma \in NC_2(l)} W_{l,F}(\pi, \sigma) \overline{\delta_\pi^F(j)} \delta_\sigma^F(i).$$

Proof. We use the fact that if $V \in \mathcal{B}(H) \otimes C(\mathbb{G})$ is a unitary representation of a compact quantum group \mathbb{G} with Haar state h , then $P_V = (\text{id} \otimes h)(V)$ is the orthogonal projection onto $\text{Fix}(V)$. Using this fact, the quantity we are interested in is the (i, j) -th matrix element of the projection $P_{U^{\oplus l}}$. Since $P_{U^{\oplus l}} = 0$ when l is odd (by Theorem 3.7), the first equality is immediate.

For the second equality, assume $l \geq 2$ is even. Let $\{\xi_\pi^F\}_{\pi \in NC_2(l)}$ be the basis for $\text{Fix}(U^{\oplus l})$ from Theorem 3.7 and define a new family $\{\tilde{\xi}_\pi^F\}_{\pi \in NC_2(l)} \subset \text{Fix}(U^{\oplus l})$ by

$$\tilde{\xi}_\pi^F = \sum_{\sigma \in NC_2(l)} W_{l,F}^{1/2}(\pi, \sigma) \xi_\sigma^F,$$

where $W_{l,F}^{1/2}$ is the matrix square root of $W_{l,F}$. Then $\{\tilde{\xi}_\pi^F\}_{\pi \in NC_2(l)}$ is an orthonormal basis for $\text{Fix}(U^{\oplus l})$ by Theorem 3.8 and $P_{U^{\oplus l}} = \sum_{\pi \in NC_2(l)} |\tilde{\xi}_\pi^F\rangle\langle\tilde{\xi}_\pi^F|$. Therefore

$$\begin{aligned} h_{O_F^+}(u_{i(1)j(1)} u_{i(2)j(2)} \cdots u_{i(l)j(l)}) &= \langle e_i | P_{U^{\oplus l}} e_j \rangle \\ &= \sum_{\rho \in NC_2(l)} \langle \tilde{\xi}_\rho^F | e_j \rangle \langle e_i | \tilde{\xi}_\rho^F \rangle \\ &= \sum_{\pi, \sigma, \rho \in NC_2(l)} W_{l,F}^{1/2}(\rho, \pi) \langle \xi_\pi^F | e_j \rangle W_{l,F}^{1/2}(\rho, \sigma) \langle e_i | \xi_\sigma^F \rangle \\ &= \sum_{\pi, \sigma \in NC_2(l)} W_{l,F}(\pi, \sigma) \langle \xi_\pi^F | e_j \rangle \langle e_i | \xi_\sigma^F \rangle \\ &= \sum_{\pi, \sigma \in NC_2(l)} W_{l,F}(\pi, \sigma) \overline{\delta_\pi^F(j)} \delta_\sigma^F(i). \end{aligned}$$

□

3.3. Integrating *-monomials over O_F^+ . Note that the defining relations for the generators $\{u_{ij}\}_{1 \leq i, j \leq N}$ of $(C(O_F^+), h_{O_F^+})$ imply that the family $\{u_{ij}\}_{1 \leq i, j \leq N}$ is self-adjoint. Moreover, taking the deformation matrix F to be in canonical form, as defined in Section 3.1, we can write

$$F = F_\rho^{(c)} = \begin{pmatrix} 0 & D_k(\rho) & 0 \\ cD_k(\rho)^{-1} & 0 & 0 \\ 0 & 0 & 1_{N-2k} \end{pmatrix}.$$

where $D_k(\rho)$ denotes the $k \times k$ diagonal matrix with diagonal entries given by a k -tuple $\rho = (\rho_i)_{i=1}^k \subset (0, 1]^k$, and $N = 2k$ if $c = -1$. Then $F^{-1} = cF$, and we easily compute from (3.2) that

$$(3.5) \quad u_{ij}^* = (cFUF)_{ij} = c \sum_{1 \leq r, s \leq N} F_{ir} F_{sj} u_{rs} = \begin{cases} cF_{i, i+k} F_{j+k, j} u_{i+k, j+k} & 1 \leq i, j \leq 2k \\ F_{i, i+k} u_{i+k, j} & 1 \leq i \leq 2k, j > 2k \\ F_{j+k, j} u_{i, j+k} & 1 \leq j \leq 2k, i > 2k \\ u_{ij} & i, j > 2k \end{cases},$$

where in the above equations we perform the additions $i \mapsto i + k, j \mapsto j + k$ modulo $2k$. Using these equations, it is easy to see that the fundamental representation $U = [u_{ij}]_{1 \leq i, j \leq N}$ of O_F^+ admits the following canonical block-matrix decomposition.

$$(3.6) \quad U = \begin{pmatrix} [u_{a,b}]_{1 \leq a, b \leq k} & [u_{a, b+k}]_{1 \leq a, b \leq k} & [u_{a,t}]_{\substack{1 \leq a \leq k \\ t \geq 2k+1}} \\ [c\rho_a^{-1} \rho_b^{-1} u_{a, b+k}^*]_{1 \leq a, b \leq k} & [\rho_a^{-1} \rho_b u_{a, b}^*]_{1 \leq a, b \leq k} & [\rho_a^{-1} u_{a, t}^*]_{\substack{1 \leq a \leq k \\ t \geq 2k+1}} \\ [u_{s,b}]_{\substack{1 \leq b \leq k \\ s \geq 2k+1}} & [\rho_b u_{s, b}^*]_{\substack{1 \leq b \leq k \\ s \geq 2k+1}} & [u_{s,t}]_{s, t \geq 2k+1} \end{pmatrix}.$$

Remark 3.10. From equation (3.6), we see that the C^* -algebra $C(O_F^+)$ is generated by the subset

$$\left(\{u_{ij}\}_{\substack{1 \leq i \leq k \\ 1 \leq j \leq N}} \cup \{u_{ij}\}_{\substack{1 \leq j \leq k \\ 2k+1 \leq i \leq N}} \cup \{u_{ij}\}_{2k+1 \leq i, j \leq N} \right) \subset \{u_{ij}\}_{1 \leq i, j \leq N}.$$

3.3.1. General *-moments over O_F^+ . Let $\epsilon \in \{1, *\}$ and $i, j \in [N]$. Using (3.6) (or equivalently (3.5)), we can find unique numbers $i_\epsilon, j_\epsilon \in [N]$ and $t_F(i, j, \epsilon) \in \mathbb{R}$ such that

$$(3.7) \quad u_{ij}^\epsilon = t_F(i, j, \epsilon) u_{i_\epsilon j_\epsilon}.$$

In particular, the computation of arbitrary *-moments in the generators $\{u_{ij}\}_{1 \leq i, j \leq N}$ can be computed using the formula of Theorem 3.9.

Proposition 3.11. *Let $l \in \mathbb{N}$, $i, j : [l] \rightarrow [N]$ and $\epsilon : [l] \rightarrow \{1, *\}$ be given. If l is odd, then*

$$h_{O_F^+}(u_{i(1)j(1)}^{\epsilon(1)} u_{i(2)j(2)}^{\epsilon(2)} \cdots u_{i(l)j(l)}^{\epsilon(l)}) = 0.$$

If l is even, then

$$\begin{aligned} h_{O_F^+}(u_{i(1)j(1)}^{\epsilon(1)} u_{i(2)j(2)}^{\epsilon(2)} \cdots u_{i(l)j(l)}^{\epsilon(l)}) &= \prod_{r=1}^l t_F(i(r), j(r), \epsilon(r)) h_{O_F^+} \left(\prod_{r=1}^l u_{i(r)\epsilon(r), j(r)\epsilon(r)} \right) \\ &= \prod_{r=1}^l t_F(i(r), j(r), \epsilon(r)) \sum_{\pi, \sigma \in NC_2(l)} W_{l,F}(\pi, \sigma) \overline{\delta_\pi^F(j_\epsilon)} \delta_\sigma^F(i_\epsilon), \end{aligned}$$

where $i_\epsilon = (i_{\epsilon(1)}(1), i_{\epsilon(2)}(2), \dots, i_{\epsilon(l)}(l))$ and $j_\epsilon = (j_{\epsilon(1)}(1), j_{\epsilon(2)}(2), \dots, j_{\epsilon(l)}(l))$.

The proof of this result is immediate.

3.3.2. *Variances of the generators of $C(O_F^+)$.* The simplest (non-zero) *-moments are the left and right covariances of the generators $\{u_{ij}\}_{1 \leq i, j \leq N} \subset C(O_F^+)$, i.e., the quantities $\langle u_{ij} | u_{kl} \rangle_L = h_{O_F^+}(u_{ij}^*, u_{kl})$ and $\langle u_{kl} | u_{ij} \rangle_R = h_{O_F^+}(u_{ij} u_{kl}^*)$. The left and right covariances can be computed using Proposition 3.11. Alternatively, we can compute these quantities using the Schur orthogonality relations

$$(3.8) \quad h_{O_F^+}(u_{ij}^*, u_{kl}) = \frac{\delta_{jl}(Q^{-1})_{ki}}{N_F}, \quad h_{O_F^+}(u_{ij} u_{kl}^*) = \frac{\delta_{ik} Q_{lj}}{N_F} \quad (1 \leq i, j \leq N),$$

where $Q = F^t \bar{F}$, $Q^{-1} = FF^*$ and $\text{Tr}(Q) = \text{Tr}(Q^{-1}) = N_F$. See for example [23] and the paragraphs following Theorem 7.2 of [17]. In particular, when $F = F_\rho^{(c)}$ is a canonical F -matrix as above, then structure of Q is simple:

$$Q = \begin{pmatrix} D_k(\rho)^{-2} & 0 & 0 \\ 0 & D_k(\rho)^2 & 0 \\ 0 & 0 & 1_{N-2k} \end{pmatrix}.$$

In particular, it follows from (3.8) that $\{u_{ij}\}_{1 \leq i, j \leq N}$ is an orthogonal system with respect to the inner products $\langle \cdot | \cdot \rangle_L$ and $\langle \cdot | \cdot \rangle_R$ induced by $h_{O_F^+}$, and a simple calculation shows that the $N \times N$ matrix of left and right variances $\Phi = [\langle u_{ij} | u_{ij} \rangle_L, \langle u_{ij} | u_{ij} \rangle_R]_{1 \leq i, j \leq N}$ has the following block-matrix decomposition (compare with the decomposition of the fundamental representation of O_F^+ given by (3.6)).

$$(3.9) \quad \Phi = \frac{1}{N_F} [(Q_{ii}^{-1}, Q_{jj})]_{1 \leq i, j \leq N} = \frac{1}{N_F} \begin{pmatrix} [(\rho_a^2, \rho_b^{-2})]_{1 \leq a, b \leq k} & [(\rho_a^2, \rho_b^2)]_{1 \leq a, b \leq k} & [(\rho_a^2, 1)]_{k \times (N-2k)} \\ [(\rho_a^{-2}, \rho_b^{-2})]_{1 \leq a, b \leq k} & [(\rho_a^{-2}, \rho_b^2)]_{1 \leq a, b \leq k} & [(\rho_a^{-2}, 1)]_{k \times (N-2k)} \\ [(1, \rho_b^{-2})]_{(N-2k) \times k} & [(1, \rho_b^2)]_{(N-2k) \times k} & [(1, 1)]_{k \times k} \end{pmatrix}$$

4. LARGE (QUANTUM) DIMENSION ASYMPTOTICS

Using our Weingarten formulas (Theorem 3.9 and Proposition 3.11), we can study their large (quantum) dimension asymptotics.

Let $F \in GL_N(\mathbb{C})$ be such that $F\bar{F} = c1$, and let $N_F = \text{Tr}(F^*F)$. We will call the number N_F the *quantum dimension* of O_F^+ . Note that we always have $N_F \geq N$. The following proposition shows that the Weingarten matrices $W_{l,F}$ associated to O_F^+ are asymptotically diagonal as the quantum dimension N_F tends to infinity. This result should be compared with Theorem 6.1 in [3].

Theorem 4.1. *For each $l \in 2\mathbb{N}$, we have (as $N_F \rightarrow \infty$) that*

$$N_F^{l/2} W_{l,F}(\pi, \sigma) = \delta_{\pi, \sigma} + O(N_F^{-1}) \quad (\pi, \sigma \in NC_2(l)).$$

Proof. According to Theorem 3.8, we have $W_{l,F} = G_{l,F}^{-1}$ and $G_{l,F}(\pi, \sigma) = c^{l/2 + |\pi \vee \sigma|} N_F^{|\pi \vee \sigma|}$. Observe that $|\pi \vee \sigma| = l/2$ if and only if $\pi = \sigma$, and $|\pi \vee \sigma| \leq l/2 - 1$ otherwise. Therefore we have the following asymptotic formula for $G_{l,F}$ (with respect to the operator norm):

$$G_{l,F} = N_F^{l/2} I + O(N_F^{l/2-1}) = N_F^{l/2} (I + O(N_F^{-1})).$$

Write $N_F^{-l/2}G_{l,F} = I + A_F$, where $\|A_F\| \leq C_l N_F^{-1}$ for some $C_l \geq 0$. Then for sufficiently large N_F , we have the absolutely convergent power series expansion

$$N_F^{l/2}W_{l,F} = (I + A_F)^{-1} = \sum_{r=0}^{\infty} (-1)^r A_F^r = I + O(N_F^{-1}) \quad (N_F \rightarrow \infty).$$

The result now follows. \square

The following proposition is a consequence of Theorem 4.1 and gives an asymptotic factorization of the normalized joint moments over O_F^+ .

Proposition 4.2. *Fix $l \in 2\mathbb{N}$ and $i, j : [l] \rightarrow [N]$. Then there is a constant $D_l > 0$ (depending only on l) such that*

$$\begin{aligned} N_F^{l/2} \left| h_{O_F^+}(u_{i(1)j(1)} u_{i(2)j(2)} \cdots u_{i(l)j(l)}) - \sum_{\pi \in NC_2(l)} \prod_{(s,t) \in \pi} h_{O_F^+}(u_{i(s)j(s)} u_{i(t)j(t)}) \right| \\ \leq \frac{D_l \max_{\pi, \sigma \in NC_2(l)} |\delta_\pi^F(j) \delta_\sigma^F(i)|}{N_F} \end{aligned}$$

Proof. Using Theorem 4.1, one can find a constant $D_l > 0$ such that

$$\sum_{\pi, \sigma \in NC_2(l)} |N_F^{l/2} W_{l,F}(\pi, \sigma) - \delta_{\pi, \sigma}| \leq \frac{D_l}{N_F}.$$

Since it also follows from Theorem 3.9 that

$$\begin{aligned} \sum_{\pi \in NC_2(l)} \prod_{(s,t) \in \pi} h_{O_F^+}(u_{i(s)j(s)} u_{i(t)j(t)}) &= \sum_{\pi \in NC_2(l)} \prod_{(s,t) \in \pi} N_F^{-1} F_{i(t)i(s)} \overline{F_{j(t)j(s)}} \\ &= N_F^{-l/2} \sum_{\pi \in NC_2(l)} \delta_\pi^F(i) \overline{\delta_\pi^F(j)}, \end{aligned}$$

we obtain

$$\begin{aligned} N_F^{l/2} \left| h_{O_F^+}(u_{i(1)j(1)} u_{i(2)j(2)} \cdots u_{i(l)j(l)}) - \sum_{\pi \in NC_2(l)} \prod_{(s,t) \in \pi} h_{O_F^+}(u_{i(s)j(s)} u_{i(t)j(t)}) \right| \\ = \left| \sum_{\pi, \sigma \in NC_2(l)} N_F^{l/2} (W_{l,F}(\pi, \sigma) - \delta_{\pi, \sigma}) \overline{\delta_\pi^F(j)} \delta_\sigma^F(i) \right| \\ \leq \frac{D_l \max_{\pi, \sigma \in NC_2(l)} |\delta_\pi^F(j) \delta_\sigma^F(i)|}{N_F}. \end{aligned}$$

\square

Using Propositions 3.11 and 4.2, we obtain a similar asymptotic factorization result for $*$ -moments.

Corollary 4.3. *Fix $l \in 2\mathbb{N}$, $\epsilon : [l] \rightarrow \{1, *\}$ and $i, j : [l] \rightarrow [N]$. Then there is a constant $D_l > 0$ (depending only on l) such that*

$$\begin{aligned} (4.1) \quad N_F^{l/2} \left| h_{O_F^+}(u_{i(1)j(1)}^{\epsilon(1)} \cdots u_{i(l)j(l)}^{\epsilon(l)}) - \sum_{\pi \in NC_2(l)} \prod_{(s,t) \in \pi} h_{O_F^+}(u_{i(s)j(s)}^{\epsilon(s)} u_{i(t)j(t)}^{\epsilon(t)}) \right| \\ \leq \frac{D_l \max_{\pi, \sigma \in NC_2(l)} |\delta_\pi^F(j_\epsilon) \delta_\sigma^F(i_\epsilon)| \prod_{r=1}^l |t_F(i(r), j(r), \epsilon(r))|}{N_F}. \end{aligned}$$

5. ASYMPTOTIC FREENESS IN O_F^+

We now arrive at the main asymptotic freeness results of this paper.

Let us fix a canonical matrix $F = F_\rho^{(c)} = \begin{pmatrix} 0 & D_k(\rho) & 0 \\ cD_k(\rho)^{-1} & 0 & 0 \\ 0 & 0 & 1_{N-2k} \end{pmatrix} \in \text{GL}_N(\mathbb{C})$ as in

Section 3.3. Recall from Remark 3.10 that the subset of (rescaled) matrix elements

$$\mathcal{S}_F = \{\sqrt{N_F}u_{ij}\}_{\substack{1 \leq i \leq k \\ 1 \leq j \leq N}} \cup \{\sqrt{N_F}u_{ij}\}_{\substack{1 \leq j \leq k \\ 2k+1 \leq i \leq N}} \cup \{\sqrt{N_F}u_{ij}\}_{2k+1 \leq i, j \leq N}$$

generates the C^* -algebra $C(O_F^+)$. In this section we show that the set \mathcal{S}_F is asymptotically free in the following sense.

Theorem 5.1. *Let $\mathcal{S} = \{y_{ij}\}_{\substack{1 \leq i \leq k \\ 1 \leq j \leq N}} \cup \{y_{ij}\}_{\substack{1 \leq j \leq k \\ 2k+1 \leq i \leq N}} \cup \{y_{ij}\}_{2k+1 \leq i, j \leq N}$ be a family of non-commutative random variables in a NCPS (A, φ) with the following properties.*

- (1) \mathcal{S} is freely independent.
- (2) For each i, j , the elements $y_{ij} \in \mathcal{S}$ and $\sqrt{N_F}u_{ij} \in \mathcal{S}_F$ have the same left and right variances, given by the matrix entries of $N_F\Phi$ in equation (3.9).
- (3) If either $i \leq k$ or $j \leq k$, then each y_{ij} is a generalized circular element.
- (4) If $2k+1 \leq i, j \leq N$, then y_{ij} is a standard semicircular element.

Then for each $l \in 2\mathbb{N}$, there is a constant $D_l > 0$ such that

$$\begin{aligned} & \left| h_{O_F^+}(\sqrt{N_F}u_{i(1)j(1)}^{\epsilon(1)} \cdots \sqrt{N_F}u_{i(l)j(l)}^{\epsilon(l)}) - \varphi(y_{i(1)j(1)}^{\epsilon(1)} \cdots y_{i(l)j(l)}^{\epsilon(l)}) \right| \\ & \leq \frac{D_l \max_{\pi, \sigma \in NC_2(l)} |\delta_\pi^F(j_\epsilon) \delta_\sigma^F(i_\epsilon)| \prod_{r=1}^l |t_F(i(r), j(r), \epsilon(r))|}{N_F}, \end{aligned}$$

for each $\epsilon : [l] \rightarrow \{1, *\}$ and $i, j : [l] \rightarrow [N]$.

Proof. Since \mathcal{S} is a free family consisting of generalized circular elements and standard semicircular elements satisfying conditions (2)-(4) above, Theorem 2.2 gives

$$\begin{aligned} \varphi(y_{i(1)j(1)}^{\epsilon(1)} \cdots y_{i(l)j(l)}^{\epsilon(l)}) &= \sum_{\pi \in NC_2(l)} \prod_{(s,t) \in \pi} \varphi(y_{i(s)j(s)}^{\epsilon(s)} y_{i(t)j(t)}^{\epsilon(t)}) \\ &= \sum_{\pi \in NC_2(l)} \prod_{(s,t) \in \pi} N_F h_{O_F^+}(u_{i(s)j(s)}^{\epsilon(s)} u_{i(t)j(t)}^{\epsilon(t)}). \end{aligned}$$

The theorem now follows from Corollary 4.3. □

5.1. Asymptotic freeness in the large dimension limit. Using Theorem 5.1, we see that the quantum groups O_F^+ provide asymptotic models for almost-periodic free Araki-Woods factors. That is, canonical generators of almost-periodic free Araki-Woods factors can be approximated in joint distribution by normalized coordinates over a suitable sequence of O_F^+ quantum groups.

To see this, let $\Gamma(H_{\mathbb{R}}, U_t)''$ be an almost-periodic free Araki-Woods factor. Then we can write $\Gamma(H_{\mathbb{R}}, U_t)'' = (z_i : i \in \mathbb{N})''$, where $(z_i)_{i \in \mathbb{N}}$ is a free family of $(1, \lambda_i)$ -generalized circular elements z_i (with $1 < \lambda_i < \infty$). To approximate the joint $*$ -distribution of $(z_i)_{i \in \mathbb{N}}$, define, for

each $k \in \mathbb{N}$,

$$F(k) = \begin{pmatrix} 0 & D_{k+1}(1, \sqrt{\lambda_1}, \dots, \sqrt{\lambda_k})^{-1} \\ -D_{k+1}(1, \sqrt{\lambda_1}, \dots, \sqrt{\lambda_k}) & 0 \end{pmatrix} \in \mathrm{GL}_{2k+2}(\mathbb{C}).$$

Theorem 5.2. *The family of non-commutative random variables $(z_{i,k})_{i=1}^k = (\sqrt{N_{F(k)}} u_{1,i+1})_{i=1}^k \subset (C(O_{F(k)}^+), h_{O_{F(k)}^+})$ converges in joint distribution as $k \rightarrow \infty$ to $(z_i)_{i \in \mathbb{N}} \subset (\Gamma(H_{\mathbb{R}}, U_t)'', \varphi_{\Omega})$.*

Proof. By construction, $(z_{i,k})_{i=1}^k$ and $(z_i)_{i=1}^k$ have the same left and right variances. By Theorem 5.1, we then have for any $l \in 2\mathbb{N}$, $\epsilon : [l] \rightarrow \{1, *\}$, $i : [l] \rightarrow \mathbb{N}$, and $k = k(i)$ sufficiently large,

$$\begin{aligned} & \left| h_{O_{F(k)}^+}(z_{i(1),k}^{\epsilon(1)} \dots z_{i(l),k}^{\epsilon(l)}) - \varphi_{\Omega}(z_{i(1)}^{\epsilon(1)} \dots z_{i(l)}^{\epsilon(l)}) \right| \\ & \leq \frac{D_l \max_{\pi, \sigma \in NC_2(l)} |\delta_{\pi}^{F(k)}(1_{\epsilon}) \delta_{\sigma}^{F(k)}((i+1)_{\epsilon})| \prod_{r=1}^l |t_{F(k)}(1, i(r) + 1, \epsilon(r))|}{N_{F(k)}}. \end{aligned}$$

Since both quantities $\max_{\pi, \sigma \in NC_2(l)} |\delta_{\pi}^{F(k)}(1_{\epsilon}) \delta_{\sigma}^{F(k)}((i+1)_{\epsilon})|$ and $\prod_{r=1}^l |t_{F(k)}(1, i(r) + 1, \epsilon(r))|$ are constant once l, i and ϵ are fixed, and $N_{F(k)} = \mathrm{Tr}(F(k)^* F(k)) \geq \mathrm{Tr}(1) = 2k + 2$, we conclude that

$$\left| h_{O_{F(k)}^+}(z_{i(1),k}^{\epsilon(1)} \dots z_{i(l),k}^{\epsilon(l)}) - \varphi_{\Omega}(z_{i(1)}^{\epsilon(1)} \dots z_{i(l)}^{\epsilon(l)}) \right| \leq \frac{\text{Constant}}{2k + 2} \rightarrow 0.$$

□

Remark 5.3. Using the same reasoning as in the proof of Theorem 5.2, it is easy to see that for the above sequence of quantum groups $(O_{F(k)}^+)_{k \in \mathbb{N}}$, the entire family of normalized generators

$$\mathcal{S}_{F(k)} = \{\sqrt{N_{F(k)}} u_{ij}\}_{1 \leq i, j \leq k} \cup \{\sqrt{N_{F(k)}} u_{i,j+k}\}_{1 \leq i, j \leq k}$$

of $C(O_{F(k)}^+)$ converges in distribution to a free family of generalized circular elements $\{y_{ij}\}_{1 \leq i, j < \infty} \cup \{w_{ij}\}_{1 \leq i, j < \infty}$ in a NCPS (A, φ) with the following left and right variances (determined by Theorem 5.1):

$$\varphi(y_{ij}^* y_{ij}) = \rho_i^2, \quad \varphi(y_{ij} y_{ij}^*) = \rho_j^{-2}, \quad \varphi(w_{ij}^* w_{ij}) = \rho_i^2, \quad \varphi(w_{ij} w_{ij}^*) = \rho_j^2,$$

where $\rho_1 = 1$ and $\rho_i = \lambda_{i-1}^{-1/2}$ for $i \geq 2$. Note also that there is a state-preserving *-isomorphism $W^*(\{y_{ij}\}_{1 \leq i, j < \infty} \cup \{w_{ij}\}_{1 \leq i, j < \infty})$ and $W^*((z_i)_{i \in \mathbb{N}}) = \Gamma(H_{\mathbb{R}}, U_t)''$, the almost-periodic free Araki-Woods factor we started with. (This isomorphism follows from [14, Theorem 6.4]).

5.2. Asymptotic freeness in finite dimensions. In Theorem 5.2, we saw that normalized generators of a suitable family of O_F^+ 's converge in distribution to free random variables as the size N of the matrices $F \in \mathrm{GL}_N(\mathbb{C})$ go to infinity. On the other hand, the general estimate of Theorem 5.1 shows that in the non-tracial setting, the rate of approximation to freeness is governed by the growth of the quantum dimension $N_F = \mathrm{Tr}(F^* F)$, and not the physical dimension N . This new phenomenon allows one to consider scenarios where $N_F \rightarrow \infty$, while the dimension N of $F \in \mathrm{GL}_N(\mathbb{C})$ is fixed. This is illustrated by the next theorem.

Theorem 5.4. Fix $k \in \mathbb{N}$ and a sequence $\rho = (\rho_1, \dots, \rho_k) \in (0, 1)^k$ and let

$$F(\gamma) = \begin{pmatrix} 0 & D_{k+1}(\rho) & 0 & 0 \\ D_{k+1}(\rho)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & \gamma^{-1} & 0 \end{pmatrix} \in GL_{2k+2}(\mathbb{C}) \quad (0 < \gamma < 1).$$

Then the subset of generators

$$\tilde{\mathcal{S}}_{F(\gamma)} = \{\sqrt{N_{F(\gamma)}} u_{ij}\}_{\substack{1 \leq i \leq k \\ 1 \leq j \leq 2k}} \subset (C(O_{F(\gamma)}^+), h_{O_{F(\gamma)}^+})$$

converges in distribution to a free family of generalized circular elements

$$\tilde{\mathcal{S}} = \{y_{ij}\}_{\substack{1 \leq i \leq k \\ 1 \leq j \leq 2k}}$$

in a NCPS (A, φ) with left and right variances given by

$$\varphi(y_{ij}^* y_{ij}) = \rho_i^2, \quad \varphi(y_{ij} y_{ij}^*) = \begin{cases} \rho_j^{-2} & 1 \leq j \leq k \\ \rho_j^2 & k+1 \leq j \leq 2k \end{cases}.$$

Remark 5.5. Note that Theorem 5.4, makes a statement about the limiting distribution of a subset $\tilde{\mathcal{S}}_{F(\gamma)}$ of generators of $C(O_{F(\gamma)}^+)$. We cannot make a statement about the asymptotic freeness of the entire family of generators $\mathcal{S}_{F(\gamma)}$ in this setting since some of these variables do not have limiting distributions. For example,

$$h_{O_{F(\gamma)}^+}(\sqrt{N_{F(\gamma)}} u_{k+1,k+1}^{\epsilon(1)} \sqrt{N_{F(\gamma)}} u_{k+1,k+1}^{\epsilon(2)}) = \begin{cases} \gamma^2 & \epsilon(1) = *, \epsilon(2) = 1 \\ \gamma^{-2} & \epsilon(1) = 1, \epsilon(2) = *, \end{cases}$$

which implies that $\sqrt{N_{F(\gamma)}} u_{k+1,k+1}$ does not have a limiting distribution as $\gamma \rightarrow 0$.

Proof of Theorem 5.4. By construction, the families $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{S}}_{F(\gamma)}$ have the same left and right variances (which are independent of γ). Therefore we may apply Theorem 5.1 to obtain, for any $l \in 2\mathbb{N}$, $\epsilon : [l] \rightarrow \{1, *\}$, $i : [l] \rightarrow [k]$, $j : [l] \rightarrow [2k]$,

$$\begin{aligned} & \left| h_{O_{F(\gamma)}^+}(\sqrt{N_{F(\gamma)}} u_{i(1)j(1)}^{\epsilon(1)} \cdots \sqrt{N_{F(\gamma)}} u_{i(l)j(l)}^{\epsilon(l)}) - \varphi(y_{i(1)j(1)}^{\epsilon(1)} \cdots y_{i(l)j(l)}^{\epsilon(l)}) \right| \\ & \leq \frac{D_l \max_{\pi, \sigma \in NC_2(l)} |\delta_\pi^{F(\gamma)}(j_\epsilon) \delta_\sigma^{F(\gamma)}(i_\epsilon)| \prod_{r=1}^l |t_{F(\gamma)}(i(r), j(r), \epsilon(r))|}{N_{F(\gamma)}}. \end{aligned}$$

Since the numerator of the above expression is fixed with respect to γ and $N_{F(\gamma)} = \gamma^2 + \gamma^{-2} + \sum_{i=1}^k (\rho_i^2 + \rho_i^{-2}) \rightarrow \infty$ as $\gamma \rightarrow 0$, the theorem follows. \square

5.3. A remark on the free unitary case. Let $c = \pm 1$ and $F \in GL_N(\mathbb{C})$ be a canonical F -matrix. In this section we briefly comment on the free unitary quantum groups U_F^+ .

In this case, it is known from the fundamental work of Banica [2] that there is an injective $*$ -homomorphism $C(U_F^+) \hookrightarrow C(\mathbb{T}) * C(O_F^+)$ (the unital free product of $C(\mathbb{T})$ and $C(O_F^+)$) given by $v_{ij} \mapsto w u_{ij}$. Here, $w \in C(\mathbb{T})$ is canonical unitary coordinate function on the unit circle \mathbb{T} . Moreover, it is known that under the above embedding, $h_{U_F^+} = (\tau * h_{O_F^+})|_{C(U_F^+)}$, where τ denotes integration with respect to the Haar probability measure on \mathbb{T} .

In other words, the variables $\{v_{ij}\}_{1 \leq i, j \leq N} \subset (C(U_F^+), h_{U_F^+})$ and $\{w u_{ij}\}_{1 \leq i, j \leq N} \subset (C(\mathbb{T}) * C(O_F^+), \tau * h_{O_F^+})$ are identically distributed. Using this observation, together with some basic

facts about free independence and the results we have already obtained on O_F^+ , we arrive at the following unitary version of Theorem 5.1, whose proof we leave as an exercise to the reader. (Note that the extra freeness inside $C(U_F^+)$ given by the above free product model yields a slightly cleaner statement than Theorem 5.1.)

Theorem 5.6. *Fix a canonical deformation matrix*

$$F = F_\rho^{(c)} = \begin{pmatrix} 0 & D_k(\rho) & 0 \\ cD_k(\rho)^{-1} & 0 & 0 \\ 0 & 0 & 1_{N-2k} \end{pmatrix} \in GL_N(\mathbb{C}),$$

and let $\mathcal{S} = \{y_{ij}\}_{1 \leq i, j \leq N}$ be a family of non-commutative random variables in a NCPS (A, φ) with the following properties.

- (1) \mathcal{S} is freely independent.
- (2) For each i, j , the elements $y_{ij} \in \mathcal{S}$ and $\sqrt{N_F} v_{ij} \in C(U_F^+)$ have the same left and right variances, given by the matrix entries of $N_F \Phi$ in equation (3.9).
- (3) Each y_{ij} is a generalized circular element.

Then for each $l \in 2\mathbb{N}$, there is a constant $D_l > 0$ such that

$$\begin{aligned} & \left| h_{O_F^+}(\sqrt{N_F} v_{i(1)j(1)}^{\epsilon(1)} \cdots \sqrt{N_F} v_{i(l)j(l)}^{\epsilon(l)}) - \varphi(y_{i(1)j(1)}^{\epsilon(1)} \cdots y_{i(l)j(l)}^{\epsilon(l)}) \right| \\ & \leq \frac{D_l \max_{\pi, \sigma \in NC_2(l)} |\delta_\pi^F(j_\epsilon) \delta_\sigma^F(i_\epsilon)| \prod_{r=1}^l |t_F(i(r), j(r), \epsilon(r))|}{N_F}, \end{aligned}$$

for each $\epsilon : [l] \rightarrow \{1, *\}$ and $i, j : [l] \rightarrow [N]$.

6. O_F^+ AND QUANTUM SYMMETRIES OF FREE ARAKI-WOODS FACTORS

In this final section we return to the free orthogonal quantum groups O_F^+ and investigate to what extent they can be regarded as quantum symmetries of free Araki-Woods factors. Inspired by the fact that the free group factors $L(\mathbb{F}_N)$ admit O_N^+ as natural quantum symmetries [8], we are interested in finding canonical families of (non-tracial) non-commutative random variables (x_1, \dots, x_N) belonging to a NCPS (A, φ) whose joint distribution is O_F^+ -invariant in the following sense.

Definition 6.1. Let (A, φ) be a NCPS and consider an N -tuple $\mathbf{x} = (x_1, \dots, x_N) \subset A$. Fix $F \in GL_N(\mathbb{C})$ and let $U = [u_{ij}] \in M_N(C(O_F^+))$ be the fundamental representation of O_F^+ . We say that \mathbf{x} has an O_F^+ -invariant joint distribution (or, that \mathbf{x} is O_F^+ -rotatable) if for any $l \in \mathbb{N}$, $i : [l] \rightarrow [N]$ and any $\epsilon : [l] \rightarrow \{1, *\}$, we have

$$(6.1) \quad \sum_{j : [l] \rightarrow N} u_{i(1)j(1)}^{\epsilon(1)} u_{i(2)j(2)}^{\epsilon(2)} \cdots u_{i(l)j(l)}^{\epsilon(l)} \varphi(x_{j(1)}^{\epsilon(1)} x_{j(2)}^{\epsilon(2)} \cdots x_{j(l)}^{\epsilon(l)}) = \varphi(x_{i(1)}^{\epsilon(1)} x_{i(2)}^{\epsilon(2)} \cdots x_{i(l)}^{\epsilon(l)}) 1.$$

We note that the existence of an N -tuple \mathbf{x} with the above O_F^+ -invariance property is connected to the existence of an *action* of the quantum group O_F^+ on the von Neumann algebra generated by \mathbf{x} .

Definition 6.2. Let \mathbb{G} be a compact quantum group with von Neumann algebra $L_\infty(\mathbb{G}) = \pi_h(C(\mathbb{G}))''$ and (extended) coproduct $\Delta_r : L_\infty(\mathbb{G}) \rightarrow L_\infty(\mathbb{G}) \overline{\otimes} L_\infty(\mathbb{G})$. Let M a von Neumann algebra.

- (1) A *left action* (or simply an *action*) of \mathbb{G} on M is a normal, injective and unital $*$ -homomorphism $\alpha : M \rightarrow L_\infty(\mathbb{G}) \overline{\otimes} M$ such that $(\iota \otimes \alpha) \circ \alpha = (\Delta \otimes \iota) \circ \alpha$. We denote the action of \mathbb{G} on M by the notation $\mathbb{G} \curvearrowright^\alpha M$.
- (2) If φ is a faithful normal state on M , we say that an action $\mathbb{G} \curvearrowright^\alpha M$ is φ -*preserving* if $(\iota \otimes \varphi) \circ \alpha = \varphi(\cdot) 1_{L_\infty(\mathbb{G})}$. Such an action has notation $\mathbb{G} \curvearrowright^\alpha (M, \varphi)$.

Remark 6.3. If $\mathbf{x} = (x_1, \dots, x_N) \subset (A, \varphi)$ is an N -tuple with an O_F^+ -invariant joint distribution, then it is easy to see that

$$\alpha(x_i) = \sum_{j=1}^N \pi_h(u_{ij}) \otimes x_j \quad (1 \leq i \leq n)$$

defines a φ -preserving action $O_F^+ \curvearrowright^\alpha (W^*(x_1, \dots, x_N), \varphi)$, where $\pi_h : C(O_F^+) \rightarrow L^\infty(O_F^+)$ is the GNS representation associated to the Haar state.

We now show that such O_F^+ -invariant non-commutative random variables exist for any F . To do this, we first need a lemma.

Lemma 6.4. *Let $Q = F^t \bar{F}$, where $F \in GL_N(\mathbb{C})$ is a canonical F matrix (so that, in particular, Q is diagonal), and let $U = [u_{ij}] \in M_N(C(O_F^+))$ be the fundamental representation of O_F^+ . Then*

$$\sum_{r=1}^N u_{ir} u_{jr}^* = \delta_{ij} 1 \quad \text{and} \quad \sum_{r=1}^N u_{ir}^* u_{jr} (Q^{-1})_{rr} = \delta_{ij} (Q^{-1})_{ii} 1 \quad (1 \leq i, j \leq N).$$

Proof. The first equality is a direct consequence of the fact that the fundamental representation U is unitary. To prove the second inequality, we use [15, Proposition 3.2.17] which shows that $\bar{U} \bar{Q}^{-1} U^t = \bar{Q}^{-1}$. Since Q is diagonal and positive definite, the result follows. \square

In the case where $F \in GL_N(\mathbb{C})$ is canonical, the way to find O_F^+ -rotatable generators of a free Araki-Woods factor is to fix a column of the fundamental representation of O_F^+ and take an N -tuple of freely independent generalized circular elements with the same left and right variances as this column (up to a common non-zero scaling factor). Again, we note that after completion of a first draft of this paper, it was pointed out to the authors that the following theorem can also be obtained as a consequence of [16, Proposition 3.1].

Theorem 6.5. *Let $F \in GL_N(\mathbb{C})$ be a canonical F matrix and let $Q = F^t \bar{F}$ with diagonal entries $(Q_{ii})_{i=1}^N$. Let $\mathbf{x} = (x_1, \dots, x_N) \subset (A, \varphi)$ be a $*$ -free family of generalized circular elements with left and right covariances given by*

$$\varphi(x_i^* x_i) = Q_{ii}^{-1}, \quad \varphi(x_i x_i^*) = 1, \quad (1 \leq i \leq N).$$

Then \mathbf{x} has an O_F^+ -invariant joint $$ -distribution. In other words, there is a φ -preserving action $O_F^+ \curvearrowright^\alpha (W^*(x_1, \dots, x_N), \varphi)$ given by $\alpha(x_i) = \sum_{j=1}^N \pi_h(u_{ij}) \otimes x_j$, where $U = [u_{ij}]$ is the fundamental representation of O_F^+ .*

Proof. We must verify (6.1) for the N -tuple \mathbf{x} , for each choice of $l \in \mathbb{N}$, $i : [l] \rightarrow [N]$, and $\epsilon : [l] \rightarrow \{1, *\}$. To start, observe that when l is odd or $|\epsilon^{-1}(1)| \neq |\epsilon^{-1}(*)|$, then both sides of (6.1) are always zero. Therefore we assume $l \in 2\mathbb{N}$ and that $|\epsilon^{-1}(1)| = |\epsilon^{-1}(*)| = l/2$.

We begin by considering the case $l = 2$, and fix $1 \leq i(1), i(2) \leq N$, $\epsilon(1) \neq \epsilon(2) \in \{1, *\}$. Then we have

$$\sum_{j(1), j(2)=1}^N u_{i(1)j(1)}^{\epsilon(1)} u_{i(2)j(2)}^{\epsilon(2)} \varphi(x_{j(1)}^{\epsilon(1)} x_{j(2)}^{\epsilon(2)}) = \sum_{j(1)=1}^N u_{i(1)j(1)}^{\epsilon(1)} u_{i(2)j(1)}^{\epsilon(2)} \varphi(x_{j(1)}^{\epsilon(1)} x_{j(1)}^{\epsilon(2)}).$$

If $\epsilon(1) = 1$ and $\epsilon(2) = *$, then $\varphi(x_{j(1)}^* x_{j(1)}^*) = 1$ and the above quantity equals

$$\sum_{j(1)=1}^N u_{i(1)j(1)} u_{i(2)j(1)}^* = \delta_{i(1), i(2)} 1 = \varphi(x_{i(1)}^* x_{i(2)}^*) 1 \quad (\text{by Lemma 6.4}).$$

If $\epsilon(1) = *$ and $\epsilon(2) = 1$, then $\varphi(x_{j(1)}^* x_{j(1)}) = Q_{j(1)j(1)}^{-1}$ and the above quantity equals

$$\sum_{j(1)=1}^N u_{i(1)j(1)}^* u_{i(2)j(1)} Q_{j(1)j(1)}^{-1} = \delta_{i(1), i(2)} \varphi(x_{i(1)}^* x_{i(1)}) 1 = \varphi(x_{i(1)}^* x_{i(2)}) 1 \quad (\text{by Lemma 6.4}).$$

In each case, we obtain

$$(6.2) \quad \sum_{j(1), j(2)=1}^N u_{i(1)j(1)}^{\epsilon(1)} u_{i(2)j(2)}^{\epsilon(2)} \varphi(x_{j(1)}^{\epsilon(1)} x_{j(2)}^{\epsilon(2)}) = \varphi(x_{i(1)}^{\epsilon(1)} x_{i(2)}^{\epsilon(2)}) 1.$$

Now let $2 < l \in 2\mathbb{N}$ and fix $\epsilon : [l] \rightarrow \{1, *\}$ and $i : [l] \rightarrow [N]$. Then we have from Theorem 2.2

$$\begin{aligned} & \sum_{j:[l] \rightarrow N} u_{i(1)j(1)}^{\epsilon(1)} u_{i(2)j(2)}^{\epsilon(2)} \cdots u_{i(l)j(l)}^{\epsilon(l)} \varphi(x_{j(1)}^{\epsilon(1)} x_{j(2)}^{\epsilon(2)} \cdots x_{j(l)}^{\epsilon(l)}) \\ &= \sum_{j:[l] \rightarrow N} u_{i(1)j(1)}^{\epsilon(1)} u_{i(2)j(2)}^{\epsilon(2)} \cdots u_{i(l)j(l)}^{\epsilon(l)} \left(\sum_{\pi \in NC_2^\epsilon(l)} \prod_{(s,t) \in \pi} \varphi(x_{j(s)}^{\epsilon(s)} x_{j(t)}^{\epsilon(t)}) \right) \\ &= \sum_{\pi \in NC_2^\epsilon(l)} \left(\sum_{j:[l] \rightarrow N} u_{i(1)j(1)}^{\epsilon(1)} u_{i(2)j(2)}^{\epsilon(2)} \cdots u_{i(l)j(l)}^{\epsilon(l)} \prod_{(s,t) \in \pi} \varphi(x_{j(s)}^{\epsilon(s)} x_{j(t)}^{\epsilon(t)}) \right). \end{aligned}$$

Fix $\pi \in NC_2^\epsilon(l)$ and consider the internal sum above. Since π is non-crossing, it contains a neighboring pair $(r, r+1)$. Applying (6.2) to the partial sum over $1 \leq j(r), j(r+1) \leq N$, we obtain

$$\begin{aligned} & \sum_{j:[l] \rightarrow [N]} u_{i(1)j(1)}^{\epsilon(1)} u_{i(2)j(2)}^{\epsilon(2)} \cdots u_{i(l)j(l)}^{\epsilon(l)} \prod_{(s,t) \in \pi} \varphi(x_{j(s)}^{\epsilon(s)} x_{j(t)}^{\epsilon(t)}) \\ &= \varphi(x_{i(r)}^{\epsilon(r)} x_{i(r+1)}^{\epsilon(r+1)}) \\ & \quad \times \left(\sum_{j:[l] \setminus \{r, r+1\} \rightarrow [N]} u_{i(1)j(1)}^{\epsilon(1)} \cdots u_{i(r-1)j(r-1)}^{\epsilon(r-1)} u_{i(r+2)j(r+2)}^{\epsilon(r+2)} \cdots u_{i(l)j(l)}^{\epsilon(l)} \prod_{(s,t) \in \pi \setminus (r, r+1)} \varphi(x_{j(s)}^{\epsilon(s)} x_{j(t)}^{\epsilon(t)}) \right). \end{aligned}$$

Repeatedly applying the same principle to this new internal sum of lower order (note that $\pi \setminus (r, r+1)$ is again non-crossing), we obtain (after a total of $l/2 - 1$ steps)

$$(6.3) \quad \sum_{j:[l] \rightarrow [N]} u_{i(1)j(1)}^{\epsilon(1)} u_{i(2)j(2)}^{\epsilon(2)} \cdots u_{i(l)j(l)}^{\epsilon(l)} \prod_{(s,t) \in \pi} \varphi(x_{j(s)}^{\epsilon(s)} x_{j(t)}^{\epsilon(t)}) = \prod_{(s,t) \in \pi} \varphi(x_{i(s)}^{\epsilon(s)} x_{i(t)}^{\epsilon(t)}) 1.$$

Therefore

$$\begin{aligned} & \sum_{j:[l] \rightarrow N} u_{i(1)j(1)}^{\epsilon(1)} u_{i(2)j(2)}^{\epsilon(2)} \cdots u_{i(l)j(l)}^{\epsilon(l)} \varphi(x_{j(1)}^{\epsilon(1)} x_{j(2)}^{\epsilon(2)} \cdots x_{j(l)}^{\epsilon(l)}) \\ &= \sum_{\pi \in NC_2^{\epsilon(l)}(l)} \prod_{(s,t) \in \pi} \varphi(x_{i(s)}^{\epsilon(s)} x_{i(t)}^{\epsilon(t)}) 1 = \varphi(x_{i(1)}^{\epsilon(1)} x_{i(2)}^{\epsilon(2)} \cdots x_{i(l)}^{\epsilon(l)}) 1. \end{aligned}$$

□

Remark 6.6. It is clear that $(W^*(x_1, \dots, x_N), \varphi) \cong (\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)'', \varphi_{\Omega})$ is a free Araki-Woods factor associated to a finite dimensional orthogonal representation $(U_t)_{t \in \mathbb{R}}$ (compare with Theorem 2.3). Moreover, it is interesting to note that the type classification for the von Neumann algebras $L^{\infty}(O_F^+)$ and $\Gamma(H_{\mathbb{R}}, U_t)''$ is the same. More precisely, if $\Gamma < \mathbb{R}_+^*$ is the subgroup generated by the eigenvalues of $Q \otimes Q^{-1}$, then both of these algebras are type II₁ when $Q = 1$, III_λ if $\Gamma = \lambda^{\mathbb{Z}}$, and type III₁ otherwise. Compare [14, Theorem 6.1] and [17, Theorem 7.1].

Remark 6.7. Theorem 6.5 only considers the case of a canonical matrix F . For generic $F \in \text{GL}_N(\mathbb{C})$ such that $F\bar{F} = c1$, recall from Section 3.1 that there is a canonical F -matrix $F_{\rho}^{(c)} \in \text{GL}_N(\mathbb{C})$ and $v \in \mathcal{U}_N$ such that $F_{\rho}^{(c)} = vFv^t$ and $O_F^+ \cong O_{F_{\rho}^{(c)}}^+$. Then $O_F^+ \curvearrowright^{\alpha_F} (\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)'', \varphi_{\Omega})$, where $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ is the free Araki-Woods factor on which $O_{F_{\rho}^{(c)}}^+$ acts in the sense of the above theorem. Indeed let $\mathbf{x} = (x_1, \dots, x_N)$ be the generalized circular system constructed in Theorem 6.5, let $\alpha_{F_{\rho}^{(c)}}$ be the corresponding action, and let $\mathbf{y} = v\mathbf{x}$. Then $W^*(\mathbf{x}) = W^*(\mathbf{y})$ and one readily checks from the defining relations that

$$W = vUv^*,$$

where $W = [w_{ij}]$ and $U = [u_{ij}]$ are the fundamental representations of O_F^+ and $O_{F_{\rho}^{(c)}}^+$, respectively. A simple calculation then shows that condition (6.1) holds with the w_{ij} 's replacing the u_{ij} 's and the y_i 's replacing the x_i 's.

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